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Toom's Model with Glauber Rates: An Exact Solution Using Elementary Methods

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Abstract A simple spatially two-dimensional stochastic cellular automaton with asymmetric coupling and synchronous updating according to Glauber rates is considered. While detailed balance is violated it is still possible to compute analytically the stationary probability distribution by elementary means. The stationary distribution can be written as a canonical equilibrium distribution of a spin system on a triangular lattice with nearest neighbour coupling. Thus, the cellular automaton shows a nonequilibrium phase transition with Ising critical behaviour.

Keywords Probabilistic cellular automata · Nonequilibrium phase transition

1 Introduction

Probabilistic cellular automata constitute a paradigmatic model class to study critical phenomena in nonequilibrium systems. It is virtually impossible to give a balanced account on such a broad subject but the reader may consult [11] for a recent overview of the topic, e.g., with regards to different notions of nonequilibrium dynamics and universality classes of lattice models. The current short note was motivated by the seminal observation that certain types of simple deterministic coupled map lattices may show properties which resemble Ising phase transitions [9]. The matter has been studied in quite detail, and finite size scaling approaches pointed towards a violation of Ising universality [8] although the results have not been entirely conclusive [12, 13]. In fact, coupled map lattices are linked to probabilistic cellular automata if a proper symbolic dynamics can be established [5] (cf. as well [6] for a completely elementary account in the context of Miller-Huse-like models), and there has been recently some interest to revisit the critical behaviour of simple probabilistic cellular automata [14].

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For coupled map lattices the issue of phase transitions is still far from being settled, cf. e.g. [1] for a recent example of a dynamical system where phase transitions have been studied rigorously. Here we are aiming at a far less ambitious goal by focusing on probabilistic cellular automata. We will consider a special case of the model introduced by Andrei Toom [15].[1] This model resembles a kinetic Ising model on a square lattice with a kind of unidirectional coupling and majority updating rule. It has been proven rigorously that such a model shows spontaneously broken symmetry,[2] the mechanism for the phase transition has been analysed in quite detail from the phenomenological point of view [2], and the techniques developed for the proof have been used recently to establish phase transitions in continuous coupled map lattices [1]. Recent numerical finite size scaling studies indicate that the Toom model displays Ising critical behaviour, although the numerical results were not entirely conclusive [14].

In this short note we will consider a special case of the Toom model, i.e., a model where the synchronous updating is performed according to Glauber rates. Such a particular model is in fact far from being novel—as stressed by the referee—, it has been mentioned—probably not for the first time—by Welberry and Miller as a model for crystal growth [16],[3] and it was then revisited by Domany in his study of equilibrium properties of Ising and Potts models [3]. Properties of this simple model, in particular with regards to phase transitions are, of course, summarised in the seminal reviews on probabilistic cellular automata [4, 7] as well. Here we recall how to solve for the stationary distribution of the master equation by analytical means. The considerations are quite basic, to some extent almost trivial, although the master equation does not obey detailed balance. In fact, the references mentioned previously mainly cover the computations shown below. In that respect the exposition may be considered being largely pedagogical. Nevertheless, the Toom model with Glauber rates constitutes a very simple dynamical model in nonequilibrium Statistical Mechanics which can be solved analytically. Thus, it constitutes one of the very few models where nonequilibrium phase transitions can be studied in an elementary way, with the exposition being accessible for a large audience, an aspect which might be worth to stress within a short note in a scientific journal.

2 The Toom-Glauber Model

Let us consider a square lattice of size $L \times L$ with periodic boundary conditions. Supply each lattice site with a spin variable $\sigma^{(\mu)} \in \{-1, 1\}$ where $\mu = (\mu_1, \mu_2)$ labels the sites. For the dynamics consider a probabilistic updating rule, i.e., a transition $\sigma^{(\mu)} \to \tau^{(\mu)}$ at lattice site μ occurs with probability $w(\sigma^{(\mu)} \to \tau^{(\mu)})$. All lattice sites are updated simultaneously per time step. Spatial coupling is introduced by the transition rates $w(\sigma^{(\mu)} \to \tau^{(\mu)})$ depending on neighbouring sites. Let $\underline{\sigma} = (\sigma^{(\mu)})$ denote the state of the system, i.e., the spin lattice. The conditional probability for obtaining in one time step a state $\underline{\tau}$ given a state $\underline{\sigma}$ can be written in terms of the local transition rates as

$$W(\underline{\tau}|\underline{\sigma}) = \prod_{\mu} w(\sigma^{(\mu)} \to \tau^{(\mu)}). \tag{1}$$

[1] The model is actually mentioned as an example on page 563 of the reference. It is, however, impossible to trace back its origin. For instance, as mentioned in the reference, N.B. Vasilyev et al. published related numerical simulations as early as 1969.

[2] The proof is contained in the original reference [15], but it could well be that the reader finds it more convenient to consult e.g. [7].

[3] As an aside: this application predates Toom's original work, the latter being of course far more general.

These conditional probabilities determine a Markov process and the time-dependent probability distribution $p_n(\underline{\sigma})$ obeys the master equation

$$p_{n+1}(\underline{\tau}) = \sum_{\underline{\sigma}} W(\underline{\tau}|\underline{\sigma}) p_n(\underline{\sigma}) = p_n(\underline{\tau}) + \sum_{\underline{\sigma}} [W(\underline{\tau}|\underline{\sigma}) p_n(\underline{\sigma}) - W(\underline{\sigma}|\underline{\tau}) p_n(\underline{\tau})]. \quad (2)$$

The time-independent solution $p_*(\underline{\sigma})$ of such an equation governs the stationary properties of the model. For a system which obeys detailed balance, i.e., for reversible probabilistic cellular automata, computation of the stationary properties is fairly straightforward. It is quite well known [4] that detailed balance normally requires a spatially symmetric coupling.

To specify the transition probabilities label by $\mu_N = (\mu_1 + 1, \mu_2)$, $\mu_E = (\mu_1, \mu_2 + 1)$, $\mu_S = (\mu_1 - 1, \mu_2)$, and $\mu_W = (\mu_1, \mu_2 - 1)$ the four nearest neighbours of a lattice site $\mu = (\mu_1, \mu_2)$. Toom's original model used a "north-east" coupling and a majority rule. To facilitate analytical calculations we keep such a coupling scheme but resort to Glauber-like transition rates, i.e., $w(\sigma^{(\mu)} \to \tau^{(\mu)})$ depends on the north and the east neighbour of site μ according to

$$\begin{aligned} w(\sigma^{(\mu)} \to \tau^{(\mu)}) &= \frac{1}{2}[1 + \tau^{(\mu)} \tanh(a_C \sigma^{(\mu)} + a_N \sigma^{(\mu_N)} + a_E \sigma^{(\mu_E)})] \\ &= \exp[\tau^{(\mu)}(a_C \sigma^{(\mu)} + a_N \sigma^{(\mu_N)} + a_E \sigma^{(\mu_E)})]/Z_\mu(\underline{\sigma}), \end{aligned} \quad (3)$$

where

$$Z_\mu(\underline{\sigma}) = 2\cosh(a_C \sigma^{(\mu)} + a_N \sigma^{(\mu_N)} + a_E \sigma^{(\mu_E)}) \quad (4)$$

denotes the normalisation. We also allow for different values of the coupling constants a_C, a_N, and a_E. Since the coupling is spatially asymmetric the master equation (2) violates detailed balance. Nevertheless, we will show that the computation of the stationary solution is still rather straightforward.

3 The Stationary Distribution

Following the idea proposed in [6] it is almost trivial to show that the stationary density can be written in terms of the normalisation (4) as

$$p_*(\underline{\sigma}) = \left(\prod_\mu Z_\mu(\underline{\sigma}) \right) \Big/ Z. \quad (5)$$

The constant Z ensures for overall normalisation of the probability density. In fact, (1) and (3) tell us that

$$\begin{aligned} \sum_{\underline{\sigma}} W(\underline{\tau}|\underline{\sigma}) \prod_\mu Z_\mu(\underline{\sigma}) &= \sum_{\underline{\sigma}} \prod_\mu \exp[\tau^{(\mu)}(a_C \sigma^{(\mu)} + a_N \sigma^{(\mu_N)} + a_E \sigma^{(\mu_E)})] \\ &= \sum_{\underline{\sigma}} \prod_\mu \exp[\sigma^{(\mu)}(a_C \tau^{(\mu)} + a_N \tau^{(\mu_S)} + a_E \tau^{(\mu_W)})] \\ &= \prod_\mu 2\cosh(a_C \tau^{(\mu)} + a_N \tau^{(\mu_S)} + a_E \tau^{(\mu_W)}). \end{aligned} \quad (6)$$

For symmetric interactions the replacement $(\mu_N) \to (\mu_S)$, $(\mu_E) \to (\mu_W)$ which has occurred on the right hand side of (6) would not matter, and such a feature is the main reason why symmetric interactions are linked with detailed balance. Here, however, the individual factors in (6) differ from (4) and it is a priori not obvious that one recovers the density (5). If we consider the logarithm of each factor expansion yields

$$\begin{aligned} &\ln[2\cosh(a_C\tau^{(\mu)} + a_N\tau^{(\mu_S)} + a_E\tau^{(\mu_W)})] \\ &\quad = \alpha + \beta_v\tau^{(\mu)}\tau^{(\mu_S)} + \beta_h\tau^{(\mu)}\tau^{(\mu_W)} + \beta_d\tau^{(\mu_S)}\tau^{(\mu_W)} \end{aligned} \tag{7}$$

since the expression is an even function of the symbolic variables and all the "higher order" contributions can be reduced to products of two spins. We therefore obtain for the sum after trivial substitutions

$$\begin{aligned} &\sum_\mu \ln[2\cosh(a_C\tau^{(\mu)} + a_N\tau^{(\mu_S)} + a_E\tau^{(\mu_W)})] \\ &\quad = \sum_\mu (\alpha + \beta_v\tau^{(\mu)}\tau^{(\mu_N)} + \beta_h\tau^{(\mu)}\tau^{(\mu_E)} + \beta_d\tau^{(\mu_N)}\tau^{(\mu_E)}) \\ &\quad = \sum_\mu \ln[2\cosh(a_C\tau^{(\mu)} + a_N\tau^{(\mu_N)} + a_E\tau^{(\mu_E)})]. \end{aligned} \tag{8}$$

Thus,

$$\prod_\mu 2\cosh(a_C\tau^{(\mu)} + a_N\tau^{(\mu_S)} + a_E\tau^{(\mu_W)}) = \prod_\mu Z_\mu(\underline{\tau}) \tag{9}$$

and (5) yields indeed, due to (6), the stationary solution of (2).

The stationary density can be written in canonical form $p_*(\underline{\sigma}) = \exp(-H(\underline{\sigma}))/Z$ where the Hamiltonian, according to (7), is given by

$$H(\underline{\sigma}) = -\sum_\mu (\beta_v\sigma^{(\mu)}\sigma^{(\mu_N)} + \beta_h\sigma^{(\mu)}\sigma^{(\mu_E)} + \beta_d\sigma^{(\mu_N)}\sigma^{(\mu_E)}) - L^2\alpha. \tag{10}$$

Using (7) the interaction constants which enter the Hamiltonian (10) are easily expressed in terms of the parameters of the transition rates (3)

$$\begin{aligned} \exp(2\alpha)\sinh(2\beta_d) &= 2\sinh(2a_E)\sinh(2a_N), \\ \exp(2\alpha)\sinh(2\beta_h) &= 2\sinh(2a_C)\sinh(2a_E), \\ \exp(2\alpha)\sinh(2\beta_v) &= 2\sinh(2a_C)\sinh(2a_N), \\ \exp(4\alpha) &= 8 + 4\sinh^2(2a_C) + 4\sinh^2(2a_N) + 4\sinh^2(2a_E) \\ &\quad + 8\cosh(2a_C)\cosh(2a_N)\cosh(2a_E). \end{aligned} \tag{11}$$

The Hamiltonian (10) describes an Ising model on a triangular lattice (cf. Fig. 1). No frustration in the interactions is possible since (11) ensures that $\beta_v\beta_h\beta_d \geq 0$. As well known [10] the phase transition occurs at

$$|\sinh(2\beta_d)\sinh(2\beta_h)| + |\sinh(2\beta_d)\sinh(2\beta_v)| + |\sinh(2\beta_h)\sinh(2\beta_v)| = 1. \tag{12}$$

If one considers, for instance, the simple case of a democratic coupling $a_C = a_N = a_E = g$ (cf. [14]) we have $\beta_d = \beta_v = \beta_h = \beta$ and the critical value is given by $\sinh(2\beta_c) = 1/\sqrt{3}$,

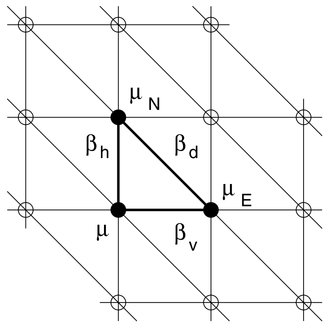

Fig. 1 Coupling scheme of the Hamiltonian (10) which determines the stationary density of the Toom-Glauber model

i.e., $\cosh^2(g_c) = 3/2$. Such an analytical result shows that the first four digits of the numerical estimate $g_c = 0.65855$ in [14] are correct.

4 Conclusion

The short calculation presented above has demonstrated that the stationary behaviour of the Toom-Glauber model can be captured by the canonical equilibrium of the nearest neighbour coupled Ising model on a triangular lattice. A priori such a feature is of course only valid in a symmetric case, i.e., without any "magnetic field". In fact, including a bias in the transition rates does not result in a simple Ising Hamiltonian with a magnetic field [2, 7]. In addition, phase diagrams of probabilistic cellular automata with bias may differ substantially from what one would expect to obtain from simple Ising Hamiltonians. Nevertheless, if one confines the considerations to symmetric systems without a magnetic field, then the current analysis gives evidence that the Toom-Glauber model supports Ising critical behaviour, in the sense that the values of the corresponding critical exponents coincide with those of the Ising model. Thus, the result is in accordance with the numerical observation that critical exponents with Ising values seem to be quite common among certain nonequilibrium models [12, 14].

It would be tempting to analyse the dynamical properties, e.g., the eigenvalue problem of (2) as well. Unfortunately, unlike the spatially one-dimensional case [6], eigenvalues and correlation functions do not seem to be accessible by straightforward methods. The analytical steps which have been used for the computation of the stationary solution cannot be generalised, apparently, to compute correlation functions or to cope with any nontrivial modification of the coupling scheme. Nevertheless, the Toom-Glauber model is a simple system where a nonequilibrium phase transition, although quite a simple one, can be studied with a completely elementary approach.

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